Abstract

An 'isomorphism' between the 'moduli space' of star products on $\mathbf{R}^2$ and 'moduli space' of all formal Poisson structures on $\mathbf{R}^2$ is established.

Quantization of Poisson Structures on $\mathbf{R}^2$

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This is a preliminary version of the paper!

The problem of quantization of Poisson structures originates from [1]. It is well known that any Poisson structure on a two-dimensional manifold is quantizable. In this paper we establish an 'isomorphism' between the 'moduli space' of star products on $\mathbf{R}^2$ and 'moduli space' of all formal Poisson structures on $\mathbf{R}^2$ by construction of a map from Poisson structures to star products. Certainly, this isomorphism follows from the Kontsevich formality conjecture [2]. Most likely, our map can be used as a first step in constructing the L_∞ quasiisomorphism in the formality conjecture for $\mathbf{R}^2$. The author would like to thank Boris Tsygan and Paul Bressler for the attention and helpful suggestions.

The set of all star-products $\mathbf{S}$ is acted upon by the group $\mathcal{D} \dot{+} \mathrm{Diffeo}\mathbf{R}^2$, where $\mathcal{D}$ is the group of operators of the form $1 + hD_1 + h^2D_2 + \ldots$ with D_k to be arbitrary differential operators. The set of all formal Poisson structures $\mathbf{P}$ consists of formal series in h with bivector fields as the coefficients. Formal Poisson structures are acted upon by the group $\mathrm{Diffeo}\mathbf{R}^2 \dot{+} \exp(h\mathrm{Vect}[[h]])$, where Vect is the Lie algebra of vector fields on $\mathbf{R}^2$. These actions define equivalence relations. We want to have a pair of maps $f_1 : \mathbf{S} \to \mathbf{P}$ and $f_2 : \mathbf{P} \to \mathbf{S}$ such that

$$f_1 \circ f_2(x) \equiv x \quad f_2 \circ f_1(x) \equiv x,$$

$$x \equiv y \to f_{1,2}x \equiv f_{1,2}y. \tag{1}$$

By a map from $\mathbf{S}$ we mean a differential expression in terms of the coefficients of the bidifferential operators corresponding to the star products. Maps from $\mathbf{P}$ are defined similarly.

We can replace $\mathbf{S}$ by a subspace. Let P, Q be a non degenerate pair of (real) polarizations of $\mathbf{R}^2$ Define a subset $\mathbf{S}_{P,Q}$ of $\mathbf{S}$ in the following

way $m \in \mathbf{S}_{P,Q}$ iff $m(f,g) = fg$ if f is constant along P or g is constant along Q.

PROPOSITION 1 *Let x, y be a nondegenerate coordinate system on $\mathbf{R}^2$ such that x is constant along Q and y is constant along P. Then there exists a unique map $\mathbf{S} \to \mathcal{D} : m \mapsto U(m) = 1 + hV(m)$ such that*

$$1) m_{P,Q}(m) = U^{-1}(m(Uf, Ug)) \in \mathbf{S}_{P,Q}$$
$$2) Ux = x,\ Uy = y,\ U1 = 1. \tag{2}$$

U is uniquely defined by the condition $U(x^{*m} * y^{*n}) = x^m y^n$ (where star denotes the star product m).

We denote by $m_{P,Q} : \mathbf{S} \to \mathbf{S}_{P,Q}$ the map which sends m to $m_{P,Q}(m)$. Further, x, y will mean the same as in Proposition 1. Thus, it is enough to find maps $p_1 : \mathbf{S}_{P,Q} \to \mathbf{P}$ and $p_2 : \mathbf{P} \to \mathbf{S}_{P,Q}$ with the same properties as f_1, f_2 have. Indeed , put $f_2 = i \circ p_2$ and $f_1 = p_1 \circ m_{P,Q}$ (here $i : \mathbf{S}_{P,Q} \to \mathbf{S}$ is the injection).

The following theorem gives an explicit construction for p_2 which appears to be a bijective map so that we can put $p_1 = p_2^{-1}$. Denote by $\mathcal{C}_P$ (resp. $\mathcal{C}_Q$) the space of functions, constant along Q (respectively P). Denote by $\mathcal{V}_P$ (resp. $\mathcal{V}_Q$) the space of vector fields preserving the polarizations and tangent to P (resp.Q). Denote by $\mathcal{D}_P$ the subalgebra of the algebra of the differential operators consisting of the operators D such that $D(\mathcal{C}_Q) \subset \mathcal{C}_Q$ and $D(fg) = fD(g)$ if $f \in \mathcal{C}_P$. Denote by $\mathcal{D}_Q$ the same algebra, where P and Q are interchanged. In the coordinates x, y we have $\mathcal{C}_P = \{f(x)\}$, $\mathcal{V}_P = \{f(x)\partial_x\}$, $\mathcal{D}_P = \sum f_i(x)\partial_x^i$ and the same things with P replaced by Q and x replaced by y. Denote by $\overline{\mathcal{D}_P}$ (respectively $\overline{\mathcal{D}_Q}$) the subring of $\mathcal{D}_P$ (respectively $\mathcal{D}_Q$) consisting of the operators which annihilate constant functions.

Note that the space of bivector fields is isomorphic to $\mathcal{V}_P \otimes_{\mathbf{R}} \mathcal{V}_Q$. Let $\mathcal{D}_{P,k}$ be the space of maps $Vp^{\otimes k} \to \overline{\mathcal{D}_P}$ (which are differential operators in terms of the coefficients).

THEOREM 1 *There exists a unique sequence $c_k \in \mathcal{D}_{P,k} \otimes \mathcal{D}_{Q,k}, k = 0, 1, 2, \ldots : c_k = \sum_i a_k^i \otimes b_k^i$, $c_0(X, Y) = 1 \otimes 1$ such that for any bivector field $\Psi = \sum_i X_i \wedge Y_i$, $X_i \in \mathcal{V}_P, Y_i \in \mathcal{V}_Q$, the formula*

$$\begin{aligned} m(\Psi, P, Q, f, g) &= fg + \textstyle\sum_{k, i_1 \ldots i_{k+1}} h^{k+1} L_{X_{i_1}}\{a_k^n(X_{i_2}, X_{i_3}, \ldots, X_{i_{k+1}})f\} \\ &\quad L_{Y_{i_1}}\{b_k^n(Y_{i_2}, Y_{i_3}, \ldots, Y_{i_{k+1}})g\} \\ &= \textstyle\sum_k m_k(f, g). \end{aligned} \tag{3}$$

gives a star-product.

Remark 1. The ansatz for the formula originates from the following observation. Given a product m from $\mathbf{S}_{P,Q}$, consider the 'set of zeros' of the $m(x,y)$. One can easily show that this set (up to 'biregular isomorphisms') is an invariant of the star product. Therefore, it is natural to require that $m(\Psi,x,y)$ would be divisible by Ψ.

To prove this theorem we need some preparation. Let us pass to the coordinates x,y. Then $X_i = \xi_i\partial_x$, $Y_i = \eta_i\partial_y$, $a_k^n(X_1,\dots,X_k)f = A_k^n(\xi_1,\dots,\xi_k,f)$, where A_n^k are polydifferential operators depending on the derivatives of ξ and f with respect to x. Similarly, $b_k^n(Y_1,\dots,Y_k,g) = B_k^n(\eta_1,\dots,\eta_k,g)$. Our task is to solve the reccurent equation

$$bm_k = -\frac{1}{2}\sum_{i=1}^{k-1}[m_i, m_{k-i}], \tag{4}$$

such that all m_k are of the form in (3). Here b is the Hochschild differential. Let us specify the meaning of the conditions imposed by (3). First, note that all m_i as well as $[m_i,m_j]$ belong to a subcomplex $K^\cdot$ of the Hochshild complex $C^\cdot(\bar{A},A)$ ($A = C^\infty(R^2)$ and $\bar{A} = A/\mathbf{R}$) such that $K^0 = K^1 = 0$; $K^i =$ polydifferential operators $D(f_1,\dots,f_i)$, such that $D(\phi(y)f_1,\dots,f_i\psi(x)) = \phi(y)\psi(x)D(f_1,\dots,f_i)$. That is, we take the cochains that only depend on $\partial_x^m f_1, \partial_y^l f_i$ and derivatives of $f_2,\dots,f_{i-1}$.

LEMMA 1 *The cohomology of $K^\cdot$ is generated over $C^\infty(R^2)$ by the class of $\partial_x\otimes\partial_y$.*

Now, let us specify exactly the space in which all m_k should be. Note that

$$m_k = \sum_{n,i_1,\dots,i_k} \xi_{i_1}\partial_x \circ A_{k-1}^n(\xi_{i_2},\dots,\xi_{i_k}) \otimes \eta_{i_1}\partial_y \circ B_{k-1}^n(\eta_{i_2},\dots,\eta_{i_k}).$$

Denote by $\mathcal{E}_P \in \mathcal{D}_{P,k}$ the space of operators of the form $D = \partial_x \circ D_1$ with $D_1 \in \mathcal{D}_{P,k}$. Certainly, $\mathcal{E}_P$ depends on a choice of the coordinate x Define $\mathcal{E}_Q$ in the same fashion.

Recall that $\Psi = \sum_i \xi_i\eta_i\partial_x\wedge\partial_y$. Put $\phi = \sum_i \xi_i\eta_i$. Then our theorem means exactly that

$$m_k = \phi K, \tag{5}$$

where $K \in \mathcal{E}_P \otimes \mathcal{E}_Q$.

Let us investigate how $\mathcal{E}_P$ and $\mathcal{E}_Q$ interact with the Hochshild differential. Let $\mathcal{A}$ be the subalgebra of functions depending on the derivatives of $\xi_1, \dots \xi_k$. Put $L_P^i = \mathcal{D}_{P,k}^{\otimes_{\mathcal{A}}^i}$ and the similar for L_Q^i. Then we have the Hochshild differential $b : L_P^i \to L_P^{i+1}$.

LEMMA 2 *The sequence*

$$0 \to \mathcal{E}_P \xrightarrow{b} L_P^2 \xrightarrow{b} L_P^3$$

is exact. This is also true if we replace P by Q.

Proof. Since the one-dimensional Hochshild complex is acyclic for dimensions bigger than 1, it suffices to show that $L_P^1 = D_{P,k}$ is equal to $\mathcal{E}_P \oplus \mathcal{D}_{P,k}^1$, where $\mathcal{D}_{P,k}^1$ is the subset of operators of order 1 from $\mathcal{D}_{P,k}$. But this splitting is given by the Euler-Lagrange operator $E : \mathcal{D}_{P,k} \to \mathcal{D}_{P,k}^1$,

$$E(\sum_{i=1}^N a_i \partial_x^i) = \left(\sum_{i=1}^N (-1)^{i-1} \partial_x^{i-1} a_i \right) \partial_x,$$

since the kernel of E is exactly $\mathcal{E}_P$.

Proof of the Theorem 1. Suppose we have found $m_1, \dots, m_{k-1}$. Show that we can solve (4) for m_k so that it is of the form in (5). Denote by A the right hand side of (4). Note that $bA = 0$ and $A \in K^3$. This means that $A = bS$, where $S \in K^2$. Note that

$$bK^2 \in L_P^2 \otimes L_Q^1 \oplus L_P^1 \otimes L_Q^2. \tag{6}$$

Let us define a projector $p_P : K^3 \to L_P^2 \otimes L_Q^1$. For this let us notice that the right hand side can be interpreted as a differential map from $\mathcal{C}_P \otimes \mathcal{C}_P \otimes \mathcal{C}_Q$ to $C^\infty \mathbf{R}^2$, (depending on $\xi_1, \dots \xi_k, \eta_1, \dots, \eta_k$), whereas the left hand side is a differential map from $\mathcal{C}_P \otimes \mathcal{C}^\infty \mathbf{R}^2 \otimes \mathcal{C}_Q$ to $C^\infty \mathbf{R}^2$. Thus, p can be defined as a restriction from $C_P \otimes \mathcal{C}^\infty \mathbf{R}^2 \otimes \mathcal{C}_Q$ to $\mathcal{C}_P \otimes \mathcal{C}_P \otimes \mathcal{C}_Q$. In the same way we can define the projector p_Q onto the second summand in (6). Thus,

$$\begin{aligned} A \; &= (p_P + p_Q)A \\ &= \sum \phi \partial_x \circ a_p^n(\xi_{i_2}, \dots, \xi_{i_p}) \circ \xi_{j_1} \partial_x \circ a_q^m(\xi_{j_2}, \dots, \xi_{j_q}) \\ &\otimes \eta_{j_1} \partial_y b_q^m(\eta_{j_2}, \dots \eta_{j_n}) \otimes \partial_y b_p^n(\eta_{i_1}, \dots, \eta_{i_p}) \\ &- \phi \partial_x a_p^n(\xi_{i_2}, \dots \xi_{i_n}) \otimes \xi_{j_1} \partial_x a_q^m(\xi_{j_1}, \dots, \xi_{j_q}) \\ &\otimes \partial_y \circ b_p^n(\eta_{i_2}, \dots, \eta_{i_p}) \circ \eta_{j_1} \partial_y \circ b_q^m(\eta_{j_2}, \dots, \eta_{j_q}). \end{aligned}$$

From this we deduce that $A = \phi T$, where $T \in L^2_P \otimes \mathcal{E}_Q \oplus \mathcal{E}_P \otimes L^2_Q$. Set $T = t_1 + t_2$, where $t_1 \in L^2_P \otimes \mathcal{E}_Q$ and $t_2 \in \mathcal{E}_P \otimes L^2_Q$. Since $bA = 0$, we see that $(b \otimes 1)t_1 = 0$. By Lemma 2, $t_1 = (bl_i) \otimes m_i$, where $l_i \in \mathcal{E}_P$, $m_i \in \mathcal{E}_Q$. Similarly, $t_2 = u_i \otimes bv_i$, where $u_i \in \mathcal{E}_P$, $v_i \in \mathcal{E}_Q$. Using the closeness of A, we have $-bl_i \otimes bm_i + bu_i \otimes bv_i = 0$. Since (by Lemma 2) $b : \mathcal{E}_P \to L^2$ is an inclusion, we have $c = l_i \otimes m_i = u_i \otimes v_i$ and $\phi bc = A$. Thus, we can put $m_k = \phi c$. This proves the existence. Let us prove the uniqueness. By Lemma 1, the ambiguity in the choice of m_k is of the form $f\partial_x \otimes \partial_y$. This should be of the form ϕK with $K \in \mathcal{E}_P \otimes \mathcal{E}_Q$, which is impossible, since all non-zero operators in $\mathcal{E}_P, \mathcal{E}_Q$ have order greater than 1. This proves the theorem.

The following corollary follows immediately from the uniqueness of the constructed star-product-product.

COROLLARY 1 *1) The map $(\Psi, P, Q) \mapsto m(\Psi, P, Q)$ is equivariant with respect to Diffeo($\mathbf{R}^2$). 2)All operators a^n_k, b^n_k are invariant with respect to the natural action of the group Diffeo($\mathbf{R}^1$).*

It looks very plausible that all the operators $a_k(X_1, X_2, \ldots, X_k)$ are just linear combinations of $L_{X_{i_1}} \ldots L_{X_{i_k}}$, where $(i_1, \ldots, i_k)$ is a permutation, and the same for b_k. This is correct at least for $k \leq 4$.

PROPOSITION 1 *The constructed map $p_1 : \mathbf{P} \to \mathbf{S}_{P,Q}$ is a bijection. The inverse map $p_2 : \mathbf{S}_{P,Q} \to \mathbf{P}$ is a well defined map.*

Proof. Consider the map $p_3 : \mathbf{S}_{P,Q} \to \mathbf{P} : \ m \mapsto \frac{1}{h}(m(x,y) - m(y,x))\partial_x \wedge \partial_y$. This is a well defined map, and it is not hard to see that p_3 is injective. Note that $p_3 \circ p_1(\psi) = \psi + O(h)$, hence, $p_3 \circ p_1$ is invertible, and we can put $p_2 = (p_3 \circ p_1)^{-1} \circ p_3$.

Let us check the properties (1). Let $D_t = \exp(tX)$ be a one-parameter local Lie group of diffeomorphisms corresponding to a vector field X. Then D_t acts naturally on $\mathbf{S}$. Using Corollary from the Theorem 1 and Proposition, we can write

$$p_1(D_t\Psi) = D_t(m(\psi, D_t^{-1}P, D_t^{-1}Q)) \equiv m_{P,Q}(m(\psi, D_t^{-1}P, D_t^{-1}Q)) = m(\chi_t, P, Q), \tag{7}$$

where $\chi_t = p_2 m(\psi, D_t^{-1}P, D_t^{-1}Q)$. Put $\delta_X(\Psi) = \frac{d}{dt}\chi_t|_{t=0}$. This is a well defined map, linear in X. Furthermore, $\delta_X = O(h)$. It is enough to prove the following.

THEOREM 2 . *There exists a linear differential operator* $A : \mathrm{Vect}[[h]] \to \mathrm{Vect}[[h]]$*, depending on* $\Psi \in \mathbf{P}$*, such that*

$$\delta_X(\Psi) = L_{A(X)}\Psi.$$

Let us explain why it is enough. First, disregarding, if needed, some terms, we can make $A(X)$ to be $O(h)$. Then (7) can be rewritten as $p_1(\Psi + tL_X\Psi) \equiv p_1(\Psi + tL_{A(x)}\Psi) + o(t)$, $p_1(\Psi + tL_{X-A(X)}\Psi) \equiv p_1(\Psi) + o(t)$. Since $A(X) = O(h)$, the equation $Y = X - A(X)$ is solvable for all Y and we deduce that $p_1(D_t\Psi) \equiv p_1(\Psi)$. Since p_1 is automatically equivariant with respect to the reflection $(x, y) \to (-x, y)$, this would mean that p_1 is equivariant with respect to the whole group of diffeomorphisms. Thus, $p_1(x) \equiv p_1(y)$. Suppose that $m_1 \equiv m_2$. Then $p_2(m_{1,2}) = p_2(\mu_{1,2})$, where $\mu_{1,2} = m_{P,Q}(m_{1,2})$. Then $\mu_2(f, g) = UD\mu_1((UD)^{-1}f, (UD)^{-1}g)$, where D is a formal diffeomorphism (that is, an element of Diffeo $\dot{+}$ exp hVect[[h]]) and U is a differential operator $U = 1 + hV$, satisfying the condition 1 of Proposition 1. Thus, $p_2(\mu_2) = p_2(D^{-1}\mu_1)$. If $D^{-1} = \exp(tX)$, and $p_2(\mu_1) = \Psi$, then $p_2\mu_2 = \chi_t \equiv \Psi$. We only need to check the equivariance with respect to the reflection $(x, y) \mapsto (-x, y)$, which immediatelyΦœ[CΦœ[Φ follows from the Corollary 1. Thus, we only need to prove Theorem 2. We need to make a reduction. Denote by $\mathcal{V}$ the space of all linear over $\mathbf{R}[[h]]$ differential operators $\mathrm{Vect}[[h]] \to \mathrm{Vect}[[h]]$, depending on $\Psi \in \mathbf{P}$. Let $\Psi = \phi\partial_x \wedge \partial_y$. Denote $\mathcal{V}_r = \overline{\mathcal{V}[\phi^{-1}]}$, where the bar means the completion in the h-adic topology.

PROPOSITION 2 *It is enough to find* A *in* $\mathcal{V}_r$*.*

Proof. Suppose we have found such an A. Let l be the least degree in h, where A has singularity. Let N be the least positive integer such that $B = \phi^N A$ does not have singularities up to h^{l+1}. Further we will write $a \equiv b$ if $\phi^{N-1}(a - b)$ does not have singularities up to h^{l+1}. Put $B = U\partial_x + V\partial_y$. Then

$$0 = L_A\Psi \equiv (1 + n)\frac{U\phi_x + V\phi_y}{\phi^N}(\partial_x \wedge \partial_y).$$

Hence,

$$B \equiv \frac{W(\phi_x\partial_y - \phi_y\partial_x)}{\phi^n} \equiv \frac{1}{N}\{\frac{W}{\phi^N}, \Psi\},$$

where $\{,\}$ is the Schouten bracket. Put $A_1 = A - \frac{1}{N}\{\frac{W}{\phi^N}, \Psi\}$. Then $L_{A_1(X)}\Psi = L_{A(X)}\Psi$ and A_1 has no singularities of order N up to h^{l+1}. Iterating this procedure, we will get rid of all the singularities.

Let us make the following reductions. Similarly to the differenrial operators on vector fields put $\mathcal{F} = \overline{\mathrm{Fun}(\mathbf{P})[\phi^{-1}]}$, where $\mathrm{Fun}(\mathbf{P})$ is the space of functions on $\mathbf{P}$. For $F \in \mathcal{F}$ put $\delta_X F(\Psi) = \frac{d}{dt}F(\Psi + t\delta_X\Psi)|_{t=0}$. Put $\omega = \frac{1}{\phi}dx \wedge dy$. Then we only need to prove that

$$\delta_X \omega = d\theta(\Psi, X).$$

for some 1-form θ. It is clear that it is enough to prove this for some form

$$\Omega = \omega + d\alpha. \tag{8}$$

Let us find a suitable form Ω. Also, we can assume that out vector field X is tangent to Q (since any vector field is a sum of a vector field tangent to P and a vector field tangent to Q).

PROPOSITION 3 *Let z be some function on R^2 such that (x, z) form a nondegenerate coordinate system. Put $\Psi = \phi\partial_x \wedge \partial_z$. For any $m = m(\Psi, P, Q)$ the differential operator $\frac{1}{h}\mathrm{ad}x$ can be represented as $\phi\partial_z(1+S(m)\circ\partial_z)$ for some well defined differential operator $S = S(\Psi, P, Q) = O(h)$.*

2) There exists a unique $f = f(x, z, P, Q, \Psi) \in \mathcal{F}$ such that

$$\phi(1 + \partial_z \circ S)f = 1. \tag{9}$$

3) Put $\Omega = \Omega(x, z, P, Q, \Psi) = f dx \wedge dz$. Then (8) holds.

Proof. 1)This immediately follows from (3). The second statement holds because $S = O(h)$. The last statement is true because (9) can be rewritten as

$$f = 1/\phi + \partial_z \tau \tag{10}$$

for some τ.

Remark. If we had an antiderivative F of f, such that $F_y = f$, then it would be $[x, F] = h$ and $\Omega = dx \wedge dF$. Thus, Ω is nothing else but the Berezin curvature [3]. Also, it is not hard to prove that our construction does not depend on a choice of z.

Now we are ready to prove the invariance. Formula (7) can be rewritten as

$$m_{P,Q_t} m(\chi_t, P, Q) = m(\Psi, P, Q_t),$$

where $Q_t = D_t^{-1} Q$. Let $A_t \circ \partial_z = 1/h\mathrm{ad}_{m(\chi_t,P,Q)} x$, $B_t \circ \partial_z = 1/h\mathrm{ad}_{m(\Psi,P,Q_t)} x$. Recall that m_{P,Q_t} is a conjugation with respect to some operator $U_t = 1 + hV_t$, and $U_t(x^n) = x^n$(see (2)). Therefore, $U_t = 1 + hW_t\partial_z$. Also, $B_t\partial_z = U_t A(t)\partial_z U_t^{-1}$ and $A_t f(\chi_t, P, Q) = B_t(f(\Psi, x, z, P, Q_t)) = 1$. One can check that $f(\Psi, x, z, P, Q_t) = f(\chi_t, P, Q) + \partial_z \circ hW_t f(\chi_t, P, Q)$. Using (10), we immediately get $\omega = \chi_t^{-1} + \partial_z \tau_t dx \wedge dz$, where $\chi_t^{-1} = (\chi_t, dx \wedge dy)^{-1} dx \wedge dy$. Therefore, $\delta_X \omega = d\alpha$, where

$$\alpha = \frac{d}{dt} d(\tau_t dx)|_{t=0}.$$

Which proves theorem 2.

References

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